BIG BANG SINGULARITY IN THE FRIEDMANN-LEMAÎTRE-ROBERTSON-WALKER SPACETIME

CRISTI STOICA

ABSTRACT. We show that the Big Bang singularity of the Friedmann-Lemaître-Robertson-Walker model does not raise major problems to General Relativity. We prove a theorem showing that the Einstein equation can be written in a non-singular form, which allows the extension of the spacetime before the Big Bang.

These results follow from our research on singular semi-Riemannian geometry and singular General Relativity [28, 29, 31] (which we applied in previous articles to the black hole singularities [32, 33, 34, 30]).

CONTENTS

INTRODUCTION

0.1. **The universe.** According to the *cosmological principle*, our expanding universe, although it is so complex, can be considered at very large scale homogeneous and isotropic. This is why we can model the universe, at very large scale, by the solution proposed by A. Friedmann [7, 9, 8]. This exact solution to Einstein's equation, describing a homogeneous, isotropic universe, is in general called the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, due to the rediscovery and contributions made by Georges Lemaître [19], H. P. Robertson [25, 26, 27] and A. G. Walker [36].

The FLRW model shows that the universe should be, at a given moment of time, either in expansion, or in contraction. From Hubble's observations, we know that the universe is currently expanding. The FLRW model shows that, long time ago, there was a very high concentration of matter, which

Date: December 11, 2011.
email: holotronix@gmail.com.
Partially supported by Romanian Government grant PN II Idei 1187.

exploded in what we call the *Big Bang*. Was the density of matter at the beginning of the universe so high that the Einstein's equation was singular at that moment? This question received an affirmative answer, under general hypotheses and considering General Relativity to be true, in Hawking's singularity theorem [10, 11, 12] (which is an application of the reasoning of Penrose for the black hole singularities [22], backwards in time to the past singularity of the Big Bang).

Of course, given that the extreme conditions which were present at the Big Bang are very far from what our observations encountered so far, and our theories managed to extrapolate, we cannot know precisely what happened then. If because some known or unknown quantum effect the energy condition from the hypothesis of the singularity theorem was not obeyed, the singularity might have been avoided, although it was a very high density. One such possibility is explored in the *loop quantum cosmology* [5, 3, 6], which leads to a Big Bounce discrete model of the universe.

We will not explore here the possibility that the Big Bang singularity is prevented to exist by quantum or other kind of effects, because we don't have the complete theory which is supposed to unify General Relativity and Quantum Theory. What we will do in the following is to push the limits of General Relativity to see what happens at the Big Bang singularity, in the context of the FLRW model. We will see that the singularities are not a problem, even if we don't modify General Relativity and we don't assume very repulsive forces which prevented the singularity.

One tends in general to regard the singularities arising in General Relativity as an irremediable problem which forces us to abandon this successful theory [15, 13, 1, 16, 2, 4]. In fact, we will see that the singularities of the FLRW model are easy to understand and are not fatal to General Relativity. In [28] we presented an approach to extend the semi-Riemannian geometry to the case when the metric can become degenerate. In [29] we applied this theory to the warped products, and provided by this ways to construct examples of singular semi-Riemannian manifolds of this type. We will develop here some ideas suggested in some of the examples presented there, and apply them to the singularities in the FLRW spacetime. We will see that the singularities of the FLRW metric are even simpler than the black hole singularities, which we discussed in [32, 33, 34, 31].

0.2. **The Friedmann-Lemaître-Robertson-Walker model of the universe.** Let's consider the 3-space at any moment of time as being modeled, up to a scaling factor, by a three-dimensional Riemannian space (Σ, g_Σ). The time is represented as an interval $I \subseteq \mathbb{R}$, with the natural metric $-\mathrm{d}t^2$. At each moment of time $t \in I$, the space Σ_t is obtaining by scaling (Σ, g_Σ) with a scaling factor $a^2(t)$. The scaling factor is therefore given by a function $a : I \to \mathbb{R}$, named the *warping function*. The FLRW spacetime is the spacetime $I \times \Sigma$ endowed with the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\Sigma^2. \tag{1}$$

It is the *warped product* between (Σ, g_Σ) and $(I, -\mathrm{d}t^2)$, with the warping function $a : I \to \mathbb{R}$.

The typical space Σ can be any Riemannian manifold we may need for our cosmological model, but because of the homogeneity and isotropy conditions, it is in general taken to be, at least at large scale, one of the homogeneous spaces S^3, $\mathbb{R}^3$, and H^3. In this case, the metric on Σ is, in spherical coordinates (r, θ, ϕ),

$$\mathrm{d}\Sigma^2 = \frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \tag{2}$$

where $k = 1$ for the 3-sphere S^3, $k = 0$ for the Euclidean space $\mathbb{R}^3$, and $k = -1$ for the hyperbolic space H^3.

0.3. **The Friedman equations.** Once we choose the 3-space Σ, the only unknown part of the FLRW metric is the function $a(t)$. To determine it, we have to make some assumptions about the matter in the universe. In general it is assumed, for simplicity, that the universe is filled with a fluid with mass density $\rho(t)$ and pressure density $p(t)$. The density and the pressure are taken to depend on t only, because we assume the universe to be homogeneous and isotropic. The stress-energy tensor is

$$T^{ab} = (\rho + p)\, u^a u^b + p g^{ab}, \tag{3}$$

where u^a is the timelike vector field ∂_t, normalized.

From the energy density component of the Einstein equation, one can derive the *Friedmann equation*

$$\rho = \frac{3}{\kappa}\frac{\dot{a}^2 + k}{a^2}, \tag{4}$$

where $\kappa := \dfrac{8\pi\mathcal{G}}{c^4}$ ($\mathcal{G}$ and c being the gravitational constant and the speed of light, which we will consider equal to 1 for now on, by an appropriate choice of measurement units). From the trace of the Einstein equation, we obtain the *acceleration equation*

$$\rho + 3p = -\frac{6}{\kappa}\frac{\ddot{a}}{a}. \tag{5}$$

From these two equations we obtain the *fluid equation*, expressing the conservation of mass-energy:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p). \tag{6}$$

Let's assume we know the function a. The Friedman equation (4) shows that we can uniquely determine ρ from a. The acceleration equation determines p from both a and ρ. Hence, the function a determines uniquely both ρ and p.

From the recent observations on supernovae, we know that the expansion is accelerated, corresponding to the existence of a positive cosmological constant Λ [24, 23]. The Friedmann's equations were expressed here without Λ,

but this doesn't reduce the generality, because the equations containing the cosmological constant are equivalent to those without it, by the substitution

$$\begin{cases} \rho & \to & \rho + \kappa^{-1}\Lambda \\ p & \to & p - \kappa^{-1}\Lambda \end{cases} \tag{7}$$

Therefore, for simplicity we will continue to ignore Λ in the following, without any loss of generality.

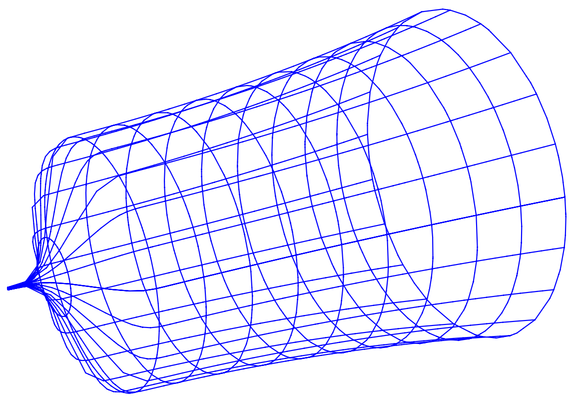

FIGURE 1. The standard view is that the universe originated from a very dense state, probably a singularity, and expanded, with a short period of very high acceleration (the inflation).

The standard view on cosmology today is that the universe started with the Big Bang, which is in general assumed to be singular, and then expanded, with a very short period of exponentially accelerated expansion, called *inflation* (Fig. 1).

1. The main ideas

The solution proposed here is simple: to show that the singularities of the FLRW model don't break the evolution equation, we show that the equations can be written in an equivalent form which avoids the infinities in a natural way. We consider useful to prepare the reader with some simple mathematical observations, which will clarify our proof. These observations can be easily understood, and combined they help us understanding the Big Bang singularity in the FLRW spacetime.

1.1. **Distance separation *vs.* topological separation.** To understand the singularities in the FLRW model, it is useful to make a parallel with another type of singularities, and their standard resolution in mathematics. Let's consider a surface in $\mathbb{R}^3$. In general it can be defined locally as the image of a map $f : U \to \mathbb{R}^3$, where $U \in \mathbb{R}^2$ is an open subset of the plane. If the function f is not injective, the surface will have self-intersections. Another way to define the surface is implicitly, as the solution of an equation.

In this case it may happen again to have self-intersections. The typical example is the cone, defined as

$$x^2 - y^2 - z^2 = 0. \tag{8}$$

We can desingularize it by making the transformation

$$\begin{cases} x &= u \\ y &= uv \\ z &= uw \end{cases} \tag{9}$$

which maps the cylinder $v^2 + w^2 = 1$ from the space parametrized by (u, v, w), to the cone from equation (8), in the space parametrized by (x, y, z). This procedure is very used in mathematics, especially in *algebraic geometry*, and it is was studied starting with Isaac Newton [20].

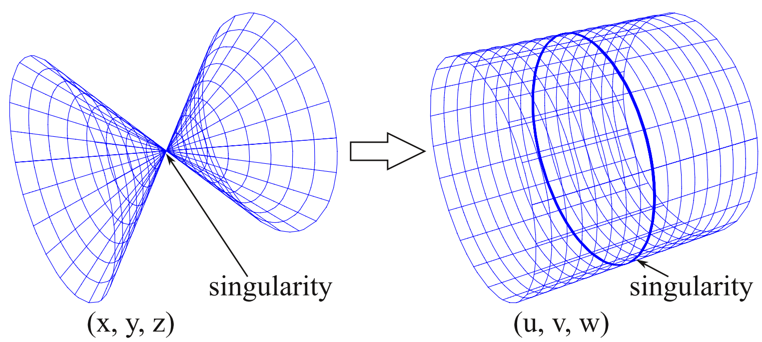

FIGURE 2. The old method of resolution of singularities shows how we can "untie" the singularity of a cone and obtain a cylinder. This illustrates the idea that it is not necessary to assume that, at the Big Bang singularity, the entire space was a point, but only that the space metric was 0.

The natural metric on the space (x, y, z) induces, by pull-back, a metric on the cylinder $v^2 + w^2 = 1$ from the space (u, v, w). The induced metric on the cylinder is singular: the distance between any pair of points of the circle determined by the equations $u = 0$ and $v^2 + w^2 = 1$ is zero. But the points of that circle are distinct.

From the viewpoint of the singularities in General Relativity, the main implication is that just because the distance between two points is 0, it doesn't mean that the two points coincide. We can see something similar even in Special Relativity: the 4-distance between two events separated by a lightlike interval is equal to 0, but those events may be distinct.

1.2. **Degenerate warped product and singularities.** The mathematics of General Relativity is a branch of differential geometry, called *semi-Riemannian (or pseudo-Riemannian) geometry* (see *e.g.* [21]). It is a generalization of the Riemannian geometry, to the case when the metric tensor

is still non-degenerate, but its signature is not positive. In this geometric framework are defined notions like contraction, Levi-Civita connection, covariant derivative, Riemann curvature, Ricci tensor, scalar curvature, Einstein tensor. These are the main ingredients of the theory of General Relativity [14, 21, 35].

The problem with the singularities is that there, these main ingredients can't be defined, or become infinite. The perfection of semi-Riemannian geometry is broken there, and by this, it is usually concluded that the same happens with General Relativity.

In [28] we introduced a way to extend semi-Riemannian geometry to the degenerate case. There is a previous approach [17, 18], which works for metric of constant signature. Our need was to have a theory valid for variable signature (because the metric changes from being non-degenerate to being degenerate), and which in addition allows us to define the Riemann, Ricci and scalar curvatures in an invariant way, and something like the covariant derivative for the differential forms and tensor fields which are of use in General Relativity. After developing this theory, introduced in [28, 31], we generalized the notion of warped product to the degenerate case, providing by this a way to construct useful examples of singularities of this nice behaved kind [29].

From the mathematics of degenerate warped products it followed that a warped product like that involved in a FLRW metric (equation 1) has only singularities which are well behaved, and which allow the extension of General Relativity to those points. At these singularities, the Riemann curvature tensor R_{abcd} is not degenerate, and it is smooth if a is smooth. The Einstein equation can be replaced by a densitized version, which allows the continuation to the singular points and avoids the infinities.

1.3. **What happens if the density becomes infinite?** In the Friedmann equations (4), (5), and (6), the variables are a, the mass/energy density ρ and the pressure density p. When $a \to 0$, ρ appears to tend to infinity, because a finite amount of matter occupies a volume equal to 0. Similarly, the pressure density p may become infinite. How can we rewrite the equations to avoid the infinities? As it will turn out, not only there is a solution to do this, but the quantities involved are actually the natural ones. As present in the equations, both ρ and p are scalar fields. But the adequate, invariant quantities actually involve the *volume element*, or the *volume form*

$$\mathrm{d}_{vol} := \sqrt{-g}\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z, \tag{10}$$

where by the factor $\sqrt{-g}$ we mean $\sqrt{-\det g_{ab}}$. The densities are in fact not the scalars ρ and p, but the quantities $\rho\mathrm{d}_{vol}$ and $p\mathrm{d}_{vol}$. They are differential 4-forms on the spacetime, and the components of these forms in a coordinate system are $\rho\sqrt{-g}$ and respectively $p\sqrt{-g}$.

Another hint that the natural quantities are the densitized ones is given by the stress-energy tensor. When it is obtained from the Lagrangian, what

we actually get is the tensor density

$$T^{ab}\sqrt{-g} = -2\frac{\delta(\mathcal{L}_M\sqrt{-g})}{\delta g_{ab}} \tag{11}$$

The values ρ and p which appear in the Friedmann equations coincide with the components of the corresponding densities only in an orthonormal frame, where the determinant of the metric equals -1, and we can omit $\sqrt{-g}$. But when $a \to 0$, an orthonormal frame would become singular, because $\det g \to 0$. A coordinate system in which the metric has the determinant -1 will necessarily be singular when $a(t) = 0$. In a non-singular coordinate system, $\det g$ has to be variable, as it is in the comoving coordinate system of the FLRW model. From (1) , the determinant of the metric in the FLRW coordinates is

$$\det g = -a^6 \det{}_3 g_\Sigma, \tag{12}$$

where by $\det_3 g_\Sigma$ we denoted the determinant of the metric tensor of the 3-dimensional typical space Σ. Since the typical space is the same for all moments of time t, $\det_3 g_\Sigma$ is constant.

Given that the metric's determinant in the comoving coordinates is

$$\sqrt{-g} = a^3\sqrt{g_\Sigma}, \tag{13}$$

which tends to 0 when $a \to 0$, we see that it might be possible for $\sqrt{-g}$ to cancel the singularity of ρ and p in $\rho \mathrm{d}_{vol}$, respectively $p\mathrm{d}_{vol}$.

2. The Big Bang singularity resolution

As explained in section §1.3, we should account in the mass/energy density and the pressure density for the term $\sqrt{-g}$.

Consequently, we make the following substitution:

$$\begin{cases} \widetilde{\rho} = \rho\sqrt{-g} = \rho a^3\sqrt{g_\Sigma} \\ \widetilde{p} = p\sqrt{-g} = p a^3\sqrt{g_\Sigma} \end{cases} \tag{14}$$

We have the following result:

Theorem 2.1. *If a is a smooth function, then the densities $\widetilde{\rho}$, $\widetilde{p}$, and the densitized stress-energy tensor $T_{ab}\sqrt{-g}$ are smooth (and therefore nonsingular), even at moments t_0 when $a(t_0) = 0$.*

Proof. The Friedmann equation (4) becomes

$$\widetilde{\rho} = \frac{3}{\kappa} a\left(\dot{a}^2 + k\right)\sqrt{g_\Sigma}, \tag{15}$$

from which it follows that if a is a smooth function, $\widetilde{\rho}$ is smooth as well.

The acceleration equation (5) becomes

$$\widetilde{\rho} + 3\widetilde{p} = -\frac{6}{\kappa} a^2\ddot{a}\sqrt{g_\Sigma}, \tag{16}$$

which shows that $\widetilde{p}$ is smooth too. Hence, for smooth a, both $\widetilde{\rho}$ and $\widetilde{p}$ are non-singular.

The four-velocity vector field is $u = \dfrac{\partial}{\partial_t}$, which is a smooth unit timelike vector. The densitized stress-energy tensor becomes therefore

$$T_{ab}\sqrt{-g} = (\widetilde{\rho} + \widetilde{p})\, u_a u_b + \widetilde{p} g_{ab}, \tag{17}$$

which is smooth, because $\widetilde{\rho}$ and $\widetilde{p}$ are smooth functions. □

Remark 2.2. *We can write now a smooth densitized version of the Einstein Equation:*

$$G_{ab}\sqrt{-g} + \Lambda g_{ab}\sqrt{-g} = \kappa T_{ab}\sqrt{-g}. \tag{18}$$

Remark 2.3. *If $a(0) = 0$, the equation* (15) *tells us that $\widetilde{\rho}(0) = 0$. From these and equation* (16) *we see that $\widetilde{p}(0) = 0$ as well. Of course, this doesn't necessarily tell us that ρ or p are zero at $t = 0$, they may even be infinite. Figure 3 shows how the universe will look, in general.*

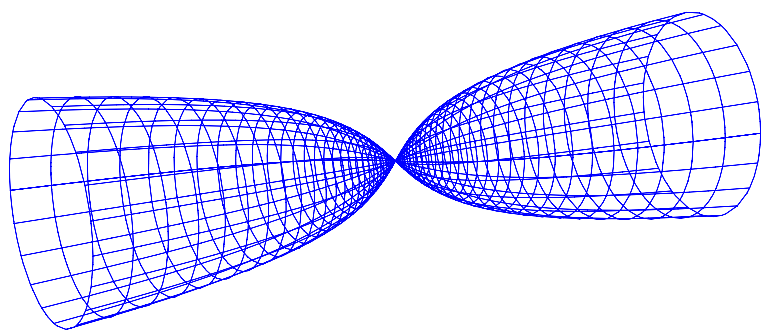

FIGURE 3. A schematic representation of a generic Big Bang singularity, corresponding to $a(0) = 0$. The universe can be continued before the Big Bang without problems.

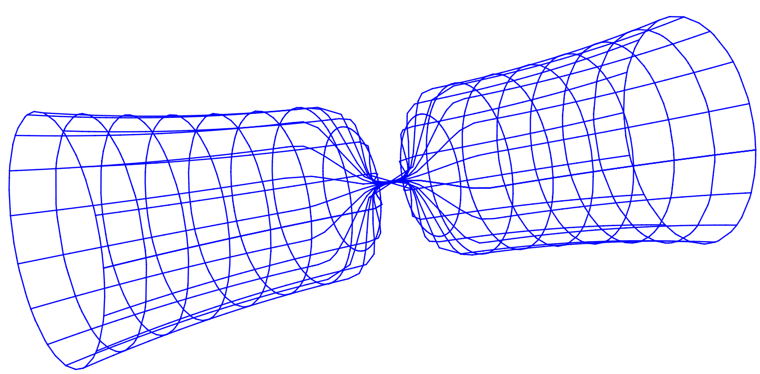

FIGURE 4. A schematic representation of a Big Bang similar to an infinitesimal Big Bounce, corresponding to $a(0) = 0$, $\dot{a}(0) = 0$, $\ddot{a}(0) > 0$.

Remark 2.4. *One interesting possibility is that when* $a(0) = 0$*, also* $\dot{a}(0) = 0$*. In this case we may have* $a(t) \geq 0$ *around* $t = 0$*, for example if* $\ddot{a}(0)$*, and obtain a Big Bang represented schematically in Fig. 4. This is very similar to a Big Bounce model, except that the singularity still appears.*

Remark 2.5. *Note that we preferred the stress-energy tensor with lowered indices,* T_{ab}*, to that with raised indices* T^{ab}*, because the former involves the smooth metric* g_{ab}*, while the latter involves its inverse,* g^{ab}*, which is singular when* g_{ab} *becomes degenerate. Similarly for the Einstein Equation. The two versions are equivalent only when the metric is non-degenerate.*

3. The evolution of the universe

This paper presented several scenarios concerning the Big Bang singularity, in the context of the Friedmann-Lemaître-Robertson-Walker model. It was found that the singularity is of degenerate type, and the time evolution is not obstructed. The solutions are schematically represented in Figures 3 and 4. These models only tell what happens at the singularity. At a global scale, the universe may re-collapse in a similar singularity and then pass again beyond it, in a cyclic cosmological model, or may expand accelerating forever, as the present day observations seem to suggest [24, 23]. Maybe the precedent universe, having $t < 0$, has no Big Bang at its origin, it just comes from the infinite past and collapses in a Big Crunch. Then, its Big Crunch becomes the Big Bang of our universe, and it starts its infinite expansion.

The cosmological arrow of our universe points from the Big Bang toward the time direction where the universe expands, which is the direction in which t increases ($t \to +\infty$). The cosmological arrow of the universe existing before the Big Bang points, of course, from the Big Bang, toward $-\infty$ (see Fig. 5).

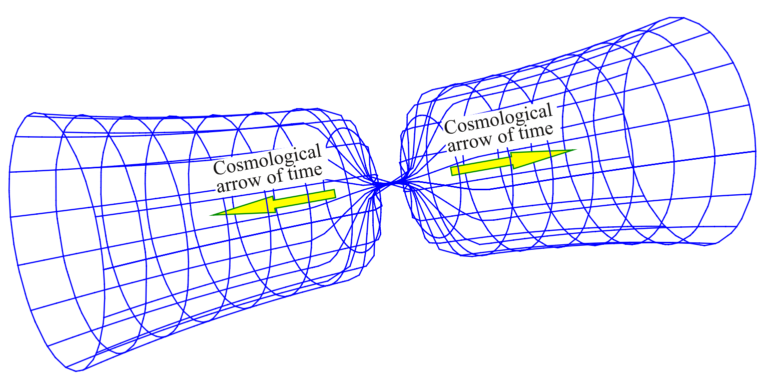

Figure 5. If the anterior universe has the cosmological arrow of time oriented toward our past, can we conclude that its entropic arrow of time also points toward our past?

It is often assumed that the entropic arrow of time is determined by some special conditions existing at the Big Bang. Of course, the "entropic arrow"

may be undefined for a simple FLRW universe, bur it may be defined in universes which are at large scale approximated by FLRW models. If the entropic arrow is determined by the cosmological arrow, then our model seems to suggest that the precedent universe has the entropic arrow of time oriented toward $-\infty$, and its time flows from the Big Bang toward $-\infty$, which is what the observers from the universe with $t > 0$ would call past.

References

1. A. Ashtekar, *Non-perturbative canonical gravity, Lecture notes in collaboration with R. S. Tate*, World Scientific, Singapore, 1991.
2. A. Ashtekar, *Singularity Resolution in Loop Quantum Cosmology: A Brief Overview*, J. Phys. Conf. Ser. **189** (2009), 012003.
3. A. Ashtekar, M. Bojowald, and J. Lewandowski, *Mathematical Structure of Loop Quantum Cosmology*, Advances in Theoretical and Mathematical Physics **7** (2003), no. 2, 233–268.
4. A. Ashtekar and E. Wilson-Ewing, *Loop quantum cosmology of Bianchi I models*, Phys. Rev. **D79** (2009), 083535.
5. M. Bojowald, *Absence of a Singularity in Loop Quantum Cosmology*, Physical Review Letters **86** (2001), no. 23, 5227–5230.
6. M. Bojowald, *Loop Quantum Cosmology*, Living Rev. Rel **8** (2005), no. 11, 2006.
7. A. Friedman, *Über die krümmung des raumes*, Zeitschrift für Physik A Hadrons and Nuclei **10** (1922), no. 1, 377–386.
8. A. Friedman, *Über die möglichkeit einer welt mit konstanter negativer krümmung des raumes*, Zeitschrift für Physik A Hadrons and Nuclei **21** (1924), no. 1, 326–332.
9. A. Friedman, *On the Curvature of Space*, General Relativity and Gravitation **31** (1999), no. 12, 1991–2000.
10. S. Hawking. The occurrence of singularities in cosmology. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 294(1439):511–521, 1966.
11. S. Hawking. The occurrence of singularities in cosmology. ii. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, pages 490–493, 1966.
12. S. Hawking. The occurrence of singularities in cosmology. iii. causality and singularities. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 300(1461):187–201, 1967.
13. S. Hawking, *Breakdown of Predictability in Gravitational Collapse*, Phys. Rev. D (1976), no. 14, 2460.
14. S. Hawking and G. Ellis, *The Large Scale Structure of Space Time*, Cambridge University Press, 1995.
15. S. Hawking and R. Penrose, *The singularities of gravitational collapse and cosmology*, Proc. Roy. Soc. London Ser. A (1970), no. 314, 529–548.
16. S. Hawking and R. Penrose, *The Nature of Space and Time*, Princeton University Press, 1996.
17. D. Kupeli, *Degenerate Manifolds*, Geom. Dedicata **23** (1987), no. 3, 259–290.
18. D. Kupeli, *Singular Semi-Riemannian Geometry*, Kluwer Academic Publishers Group, 1996.
19. G. Lemaître, *Un univers homogène de masse constante et de rayon croissant rendant compte de la vitesse radiale des nébuleuses extra-galactiques*, Annales de la Societe Scietifique de Bruxelles **47** (1927), 49–59.
20. I. Newton, *Letter to Oldenburg (24 October 1676)*, 1960, pp. 126–127.

21. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Pure Appl. Math. (1983), no. 103, 468.
22. R. Penrose, *Gravitational Collapse and Space-Time Singularities*, Phys. Rev. Lett. (1965), no. 14, 57–59.
23. S. Perlmutter, G. Aldering, G. Goldhaber, RA Knop, P. Nugent, PG Castro, S. Deustua, S. Fabbro, A. Goobar, DE Groom, et al., *Measurements of* Ω *and* Λ *from 42 High-Redshift Supernovae*, The Astrophysical Journal **517** (1999), 565.
24. A.G. Riess, A.V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, P.M. Garnavich, R.L. Gilliland, C.J. Hogan, S. Jha, R.P. Kirshner, et al., *Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant*, The Astronomical Journal **116** (1998), 1009.
25. H.P. Robertson, *Kinematics and World-Structure*, The Astrophysical Journal **82** (1935), 284.
26. H.P. Robertson, *Kinematics and World-Structure II.*, The Astrophysical Journal **83** (1936), 187.
27. H.P. Robertson, *Kinematics and World-Structure III.*, The Astrophysical Journal **83** (1936), 257.
28. C. Stoica, *On Singular Semi-Riemannian Manifolds*, arXiv:math.DG /1105.0201 (2011), 36.
29. C. Stoica, *Warped Products of Singular Semi-Riemannian Manifolds*, arXiv:math.DG /1105.3404 (2011), 23.
30. C. Stoica, *Globally Hyperbolic Spacetimes With Singularities*, arXiv:math.DG /1108.5099 (2011), 18.
31. C. Stoica, *Cartan's Structural Equations for Degenerate Metric*, arXiv:math.DG /1111.0646 (2011), 7.
32. C. Stoica, *Schwarzschild Singularity is Semi-Regularizable*, arXiv:gr-qc /1111.4837 (2011), 8.
33. C. Stoica, *Analytic Reissner-Nordstrom Singularity*, arXiv:gr-qc /1111.4332 (2011), 11.
34. C. Stoica, *Kerr-Newman Solutions with Analytic Singularity and no Closed Timelike Curves*, arXiv:gr-qc /11117082 (2011), 9.
35. Robert M. Wald, *General Relativity*, University Of Chicago Press, June 1984.
36. A.G. Walker, *On Milne's Theory of World-Structure*, Proceedings of the London Mathematical Society **2** (1937), no. 1, 90.